Modal Approach for Sound Generation in Corrugated Pipes: A Phenomenological Model

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Abstract. The sound generation in a corrugated pipe is studied based on a phenomenological approach relying on the feedback mechanism provided by the van der Pol type of governing equations. Necessary basics of vortex sound theory are presented. Effects of low Mach number flows and missing corrugations are simulated as well as different mean flow velocity pulses. The discussion aims for connecting the model features and parameters to the published experimental behavior of such pipes.

Keywords
vortex sound, corrugated pipes, Van der Pol equation, dynamical systems

1. Introduction
Corrugated pipes are inseparable part of many devices ranging from simple foot pumps to vast engineering installations containing flexible risers. The decisive structural advantage of the corrugated pipes is local rigidity still allowing global flexibility. The acoustical study focuses on the corrugation pipes drawback: fluid flowing past the corrugations causes sound production. The vortex sound source is not of a particular power per se but the mode locking with pipe resonance can result to sound of considerable amplitude causing unwanted noise or even structural problems. [3]

There are multiple ways to study this phenomenon including simplified acoustical approach as well as costly CFD simulations. In the following text we make use of a previously proposed phenomenological model relying on modeling the pressure vortex sources as forced van der Pol oscillators [1].

2. Theory
2.1. Vortex Sound Theory Requirements
Since the full description of the flow field in the corrugated pipe would unduly encumber the text only a brief review of the major features will be given and the reader is referred to existing literature for further commentary.

Sound is generated or absorbed in corrugated pipes due to the unstable flow patterns of nonzero vorticity and by the hydrodynamic reactions to such instabilities by the pipe walls (Fig. 1). We begin our quick analysis with the Helmholtz decomposition of the flow velocity field \( \mathbf{v} \):

\[
\mathbf{v} = \nabla (\phi_0 + \phi') + \nabla \times \mathbf{A} ,
\]

where the \( \phi \) and \( \mathbf{A} \) are scalar and vector potentials respectively. Mean value of the scalar potential is denoted with 0, the prime denotes perturbations of the mean scalar potential field.

According to Howe [5, 8] the time-averaged sound power \( \langle P_{vor} \rangle \) generated or absorbed by the airflow in the low Mach number limit in a control volume \( V \) could be expressed as:

\[
\langle P_{vor} \rangle = - \left\langle \int_V \rho_0 (\omega \times \mathbf{v}) \cdot \mathbf{u}_{ac} \, dV \right\rangle ,
\]

where \( \omega = \nabla \times \mathbf{v} \) is the vorticity and \( \mathbf{u}_{ac} \) the acoustic velocity defined as the unsteady part of the potential velocity field (\( \mathbf{u}_{ac} = \nabla \phi' \)).
Three important features follow from these considerations: there is a feedback-loop between the acoustic and convective flow fields, the sound should be generated predominantly in regions of higher acoustic velocity and the flow can be acoustically generative as well as dissipative, based on the interplay among the flow velocity, the vorticity and the acoustic velocity. See e.g. [5] for more details. Hence the use of the van der Pol type of equations known to have a stable limit cycle resulting from strengthening of weak disturbances and attenuating the strong ones.

### 2.2. Model Equations

Let $U$ denote a mean flow assumed for simplicity to be uniform and time independent through a corrugated pipe of length $L$. The pipe length is assumed to be much bigger than the pipe cross-section so the system should be deemed one dimensional. The van der Pol type of equation is used to model an elementary source $P_n$ located at $n$-th corrugation at a stable point $x_n$ [1]:

$$\frac{d^2 P_n}{dt^2} + A \omega_S \left[ \left( \frac{P_n}{BU^n} \right)^2 - 1 \right] \frac{dP_n}{dt} + \omega_S^2 P_n = C \frac{\partial p}{\partial x} \Big|_{x=x_n},$$

(3)

where $A$, $B$, $C$ and $\alpha$ are parameters described below, $\omega_S$ is the Strouhal angular frequency and $p = p(x,t)$ is the acoustic pressure inside the pipe. Note that the evaluation of acoustic pressure spatial derivative gives (with proper constant) an approximation of the local acoustic velocity.

The acoustic pressure $p$ is governed by a convective wave equation driven by an external force by unit volume $f_{src,x}$:

$$\left( \frac{D^2}{Dt^2} - c_0^2 \frac{\partial^2}{\partial x^2} \right) p = -\frac{\partial f_{src,x}}{\partial x}. \quad \text{(4)}$$

We identify $U$ as the unidimensional convective flow velocity for the wave equation so the operator $\frac{D^2}{Dt^2}$ yields $\frac{\partial^2}{\partial t^2} + U \frac{\partial^2}{\partial x^2} + U^2 \frac{\partial^2}{\partial t^2}$. The external force per unit volume is composed of the elementary sources contributions. Making use of the Dirac delta function $\delta$ sampling property it gives:

$$f_{src,x}(x,t) = \sum_{n=1}^{N} P_n(t) \delta(x-x_n), \quad \text{(5)}$$

where $N$ is the number of corrugations. Combining the last relations we arrive to the final form of the wave equation:

$$\frac{\partial^2 p}{\partial t^2} + U \frac{\partial^2 p}{\partial x \partial t} + (U^2 - c_0^2) \frac{\partial^2 p}{\partial x^2} = \sum_{n=1}^{N} P_n(t) \frac{\partial}{\partial x} \delta(x-x_n). \quad \text{(6)}$$

Considering the time-harmonic solution to the homogeneous convective wave equation gives the convective Helmholtz equation for the acoustic pressure. For simplicity, we consider the ideally reflecting pipe ends so the wave number has values of $\frac{k \pi}{L}$ where $k$ is an integer and the spatial modal shapes $\Phi_k(x)$ are:

$$\Phi_k(x) = \sin \frac{k \pi x}{L}. \quad \text{(7)}$$

We are aiming to express the acoustic pressure as a sum of orthogonal modes truncated at an appropriately chosen $K$:

$$p(x,t) = \sum_{k=1}^{K} q_k(t) \Phi_k(x). \quad \text{(8)}$$

Using the orthogonality of the modes we arrive to the following equation for modal amplitudes $q(t)$ (see e.g. [6], p. 189 for a detailed decomposition procedure):

$$\ddot{q}_k + 2k \zeta \omega_k q_k + k^2 \left( \omega^2 - \frac{\pi^2 U^2}{L^2} \right) q_k = 2 \frac{k \pi \omega}{L} \sum_{n=1}^{N} P_n(t) \cos \frac{k \pi x_n}{L}, \quad \text{(9)}$$

where the dots denote time derivatives as usual and the second term in the left hand side represents an ad hoc added damping (see below). The pressure vortex sources equation (3) becomes:

$$\frac{d^2 P_n}{dt^2} + A \omega_S \left[ \left( \frac{P_n}{BU^n} \right)^2 - 1 \right] \frac{dP_n}{dt} + \omega_S^2 P_n = \pi C \sum_{k=1}^{K} k q_k(t) \cos \frac{k \pi x_n}{L}. \quad \text{(10)}$$

The Eqs. (9), (10) presents a default set for further work. We make the equations nondimensional by introducing a scaled time $\tau = \omega t$ where $\omega = \frac{\pi c_0}{L}$, the Mach number $M = \frac{U}{c_0}$ and the Strouhal angular frequency:

$$\omega_S = \frac{1}{2 \pi} Sr \frac{U}{\Lambda}, \quad \text{(11)}$$

where $Sr$ is the Strouhal number and $\Lambda$ is a characteristic length (see below).

This procedure yields a new set of coupled ordinary differential equations:
et al. [2] connects the coefficient $A$ linear damping and the limit cycle size respectively. Popescu et al. [2] connects the coefficient $A$ with the ratio of boundary layer thickness to the pipe diameter. Hence the numerical value of the coefficient should be much less than one, since in fully turbulent flow the feedback mechanism does not lock and therefore the singing phenomenon would not take place. The characteristic distance of Strouhal law $\Lambda$ was originally taken as the length of a corrugation pitch. Nevertheless, the recent literature shows that extending this distance by the radius of the upstream edge leads to more realistic results [5]. For the sake of simplicity we hold the Strouhal number constant ($Sr = 0.4$).

\[
\begin{aligned}
\ddot{P}_n + \nu \dot{P}_n &= \frac{P_n^2}{B^2 \pi^2 M^{2 \alpha}} - 1 \dot{P}_n + \nu^2 P_n \\
\ddot{q}_k &= 2k \zeta \dot{q}_k + k^2 (1 - M^2) q_k \\
\dot{q}_k &= \frac{2k}{\pi} \sum_{n=1}^{N} P_n(\tau) \cos \frac{k \pi x_n}{L},
\end{aligned}
\]

where $\nu = 2ML^2 Sr$. Note that the coefficient $(1 - M^2)$ on the right hand side of the modal amplitudes equation is the only consequence of using the convective wave equation.

\[\frac{\dot{P}_n + \nu \dot{P}_n}{P_n^2} = \frac{1}{B^2 \pi^2 M^{2 \alpha}} - 1 \dot{P}_n + \nu^2 P_n \]

\[\ddot{q}_k + 2k \zeta \dot{q}_k + k^2 (1 - M^2) q_k = \frac{2k}{\pi} \sum_{n=1}^{N} P_n(\tau) \cos \frac{k \pi x_n}{L},\]

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2.3. Meaning of the Coefficients

The proposed model contains various coefficients which should be shortly examined in the physical context now.

Behavior of the van der Pol oscillator is governed by the coefficients $A$ and $B$ which controls the strength of nonlinear damping and the limit cycle size respectively. Popescu et al. [2] connects the coefficient $A$ with the ratio of boundary layer thickness to the pipe diameter. Hence the numerical value of the coefficient should be much less than one, since in fully turbulent flow the feedback mechanism does not lock and therefore the singing phenomenon would not take place. The physical meaning of the parameter $B$ is still missing. Its value is believed to be connected with the resonant effect of the corrugation cavity and its interaction with the cavity trailing edge.

The coupling coefficient $C$ is reported [2] to be connected with a ratio of sounding frequency to the eigenfrequency of the single corrugation (usually approximated as a Helmholtz resonator). Therefore its value should depend on the mean flow velocity provided that Strouhal law is a valid approximation. In this article the relation $C = 10^4 M^2$ is used.

The coefficient $\alpha$ presents possible generalization of the system enhancing its capabilities to fit on the experimental data. The value of $\alpha$ should be near 2 and this particular value is used throughout this article. The coefficient $\zeta$ accounts for acoustic losses, its value being small enough for the modes to be considered real over the pipe length. The characteristic distance of Strouhal law $\Lambda$ was originally taken as the length of a corrugation pitch. Nevertheless, the recent literature shows that extending this distance by the radius of the upstream edge leads to more realistic results [5].

3. Numerical Results

The linear stability analysis shows that the origin of the dynamical system is an unstable focus for any physically meaningful coefficient values (see e.g. [7]). Therefore a limit cycle will be reached whenever any of the initial conditions is nonzero. We hold all the acoustic initial values to be zero and all the pressure source variables to be of $O(10^{-4})$ in a random distribution (variable over the presented trials). Beside the above mentioned coefficients the rest is defined as follows: $A = 0.005$, $B = 0.0001$, $\alpha = 2$, $\zeta = 0.001$. Length of the pipe was 0.4 m with 50 corrugations and 10 computed acoustic modes. The characteristic distance $\Lambda$ was 6 mm, the sound speed 340 m·s$^{-1}$. The classical Runge-Kutta 4th order scheme was used on Eqs. (12), (13).

The presented data were obtained from eq. (8) (for $x$ at the open end) and subsequently analysed by means of the Hilbert transformation to obtain the instantaneous amplitude and frequency. In order to ease the comprehension the data were low-pass filtered before the visualization. The reference value for the SPL computation was $2 \cdot 10^{-5}$ Pa.

3.1. Velocity sweeps

The first scenario simulates the system subjected to a linear velocity sweep implemented as the growing Mach number.

Effect of Low Mach Number Flow

Fig. 2 shows data obtained by solving the equations without any exception (the green curve, referred to as ”reference run”). The red curve represents the solution without the Mach number term in the Eq. (13), i.e. the solution neglecting the flow effects. It is apparent that the difference between the simulations becomes mildly noticeable with the growing Mach number but the flow effect is very weak, essentially negligible in a qualitative point of view.

Effect of Missing Corrugations

The previous commentary and Eqs. (12), (13) implies that the sound is mainly generated in the regions of acoustic velocity anti-nodes. Fig. 3 shows a system with two corrugations missing at each end of the pipe simulating smooth inlet and outlet regions. All the acoustic modes have a velocity antinode at these regions and therefore the loss of power is spread rather equally among them.

In the subsequent case (Fig. 4) only the corrugations corresponding to the 3rd mode velocity antinodes are removed which results in attenuation of this mode.
3.2. Velocity pulses

Simulation of the system subjected to three gaussian shaped velocity pulses is presented on Fig. 5. The dimensionless time corresponds to the number of Runge-Kutta algorithm steps (referred to as "RK4 steps" in the figure). Note the asymmetry of the sounding frequency profile due to the mode locking which may result in missing of some of the modes.

4. Discussion & Conclusions

Higher Mach number values are inaccessible without further careful analysis because the "non-acoustic" flows could not be deemed incompressible any more and the relatively simple Howe’s formula (2) would not hold. Beside that, the radiation losses due to the convection of acoustic waves outside the tube are of \(O(M)\) but the effects on the pipe eigenfrequencies is of \(O(M^2)\) [4]. Therefore the presented model is valid only in the low Mach number limit. It has been proven that in such limit the influence of convection on the nonlinear dynamical system is negligibly low.

Using the nontrivial velocity build-up (sweep or pulse) is necessary for system to behave realistic. If a set of different Mach numbers is examined assuming that their value is reached immediately, the frequency lock-in plateaus does not appear which is inconsistent with experiments.

Weakness or total absence of the first acoustic mode (see e.g. Fig. 2) is caused by the lacking feedback. In practice this is assumed to be a result of the flow insufficient receptivity at lower Reynolds numbers [4].

It is worth to note that the ratio \(\frac{L}{\Lambda}\) appearing in the scaled pressure vortex source eigenfrequency \(\nu = 2M\frac{L}{\Lambda}Sr\) is in order proportional to the number of corrugations \(N\), precise value being dependent on a specific design of the pipe through the trailing edge radius and corrugations spacing. This feature is promising for a possible future fitting the phenomenological model on the measured data.

The effect of corrugations compliance on the sound transmission could be accounted by introducing the effective sound speed \(c_{\text{eff}}\) [4]:

\[
c_{\text{eff}} = c_0 \sqrt{\frac{V_i}{V_p}},
\]

where \(V_i\) is the volume of a corrugation segment defined by its minimal inner diameter and \(V_p\) the volume of the whole corrugation segment. It is straightforward that \(c_{\text{eff}} < c_0\). Nevertheless, a possible transformation \(c_0 \rightarrow c_{\text{eff}}\) in the wave equations would not qualitatively change the presented system behavior.

Note that the presented SPL corresponds to the acoustic pressure in the pipe, not to the radiated pressure, which would be considerably lower.

There are multiple ways to be followed in a future research. The internal pressure amplitudes ask for a nonlin-
ear finite-amplitude formulation of the resonator describing wave equation. Other possibilities include analytical solution of the proposed set of equations or accounting for the radiation losses.

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### References


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