Comparison of measured and calculated value of mutual inductance of two helical coils

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Abstract. In this article was compared the mutual inductance which was calculated, measured on a physical sample. The mutual inductance is important parameter for a wireless energy transfer because it gives us information about how many energy can be transferred through an the certain distance between the two helical coils. The dependency between the distance between the helical coils is direct proportional to the mutual inductance it means that for achievement a high efficiency is necessary achieve a high value of a mutual inductance.

Keywords mutual inductance; helical coils; wireless energy transfer.

1. Introduction

One of the important parameter for wireless power transmission is the mutual inductance that tells us about how much is the electromagnetic coupling between two cylindrical coils strong, when an air is positioned between them, and when are without coaxial and angular deflection.

Mutual inductance between two coils originates from the magnetic flux, which is generated by one of the coils. For the sake of simplicity, let imagine two coils with one turn, whereby such a coil can be defined as a loop. Transmitting loop generates magnetic flux \( \Phi_1 \) and electromagnetic force \( \varepsilon \). These variables are result of time-varying current flowing through transmitter. A part of generated flux \( \Phi_1 \) is enveloped by receiving coil. This flux is designated as \( \Phi_{12} \) and is defined as mutual flux. Mutual flux for given area of coil generates magnetic induction, which is given by Biot-Savavarth-Laplace law.

Mutual magnetic flux \( \Phi_{12} \), which flows through the area of receiving coil \( S \) can be defined as follows:

\[
\Phi_{12} = \int_{S_2} B_2 dS_1 = \left( \frac{\mu_0}{4\pi} \int_{S_{1,2}} \frac{dl_1 \times r_{12}}{r_{12}^2} dS_1 \right) I_2
\]

In the same way the magnetic flux \( \Phi_{21} \) can be defined and as valid for the reverse flow (i.e. receiving coil act as source).

\[
\Phi_{21} = \int_{S_1} B_1 dS_2 = \left( \frac{\mu_0}{4\pi} \int_{S_{1,2}} \frac{dl_2 \times r_{12}}{r_{12}^2} dS_2 \right) I_1
\]

Formulas (1) and (2) can be further simplified into formulas (3) and (4):

\[
\Phi_{21} = L_{12} I_2
\]
\[
\Phi_{21} = L_{21} I_1
\]

Consequently for the calculation of mutual inductances, next formulas are valid:

\[
L_{12} = \frac{\mu_0}{4\pi} \int_{S_{1,2}} \frac{dl_2 \times r_{12}}{r_{21}^2} dS_1
\]
\[
L_{21} = \frac{\mu_0}{4\pi} \int_{S_{1,2}} \frac{dl_1 \times r_{12}}{r_{12}^2} dS_2
\]

Inductances \( L_{12} \) and \( L_{21} \) are the same. This fact can be confirmed, when we express magnetic induction \( B \) in the way of vector potential [7]. Based on this, it can be stated that \( L_{12} = L_{21} = M \), whereby \( M \) is defined as mutual inductance, whose static formula is as follows:

\[
M = \frac{\Phi_{21}}{I_1} = \frac{\Phi_{12}}{I_2}
\]

The meaning of several symbols and variables is interpreted on Fig. 1.
Mathematical calculation

The first step before mathematical calculation has been verified was development of source code in Matlab. This code serves for computation of self-inductance, and mutual inductances. Main parameters of coils were taken over from existing system for wireless energy transfer.

For calculation of the magnetic flux for round coil use of vector’s potential $A$ is better choice, instead of magnetic induction $B$. Vector’s potential is defined by

$$ B = \text{rot} A $$

Then magnetic flux can be defined by integration of the vector’s potential along circumference of the current turn.

$$ \Phi = \oint_A A \, dl $$

Advantage of using vector’s potential is that, we can work with it’s as a scalar quantity, for a coaxial arrangement of the round coils. Vector’s potential driven by thin current turn can be written as

$$ A_v(r, \varphi, z) = \frac{I}{4\pi} \int_0^{2\pi} \frac{\cos \varphi'}{\sqrt{r^2 + R_1^2 - \cos \varphi' + (z-z')^2}} \, d\varphi' $$

For better explanation of (10), see [1]. Magnetic flux of two round wires are shown in (fig. 2), and can by written as:

$$ \Phi = \frac{I}{2} \int_{R_1}^{R_2} \frac{\cos \varphi'}{\sqrt{R_1^2 + R_2^2 - 2R_1 R_2 \cos \varphi' + (z-z')^2}} \, d\varphi' $$

We can define Magnetic flux for two coaxial round thin coils as integral of the magnetic flux for two wires over whole height of the first coil, while elementary part of coil on the position $z' \in [z_{11}, z_{12}]$ (fig. 3) has number of turns:

$$ \frac{N_1}{h_1} \, dz' . $$

Then it integrate over whole height of the second coil, while elementary part of coil on the position $z \in [z_{21}, z_{22}]$ has number of turns:

$$ \frac{N_2}{h_2} \, dz . $$

Magnetic flux for two coaxial thin round coils can by written as:

$$ \Phi \overset{C/C}{=} \frac{\mu_0 N_1 N_2}{2 h_1 h_2} R_1 R_2 \int \frac{\cos \varphi'}{\sqrt{R_1^2 + R_2^2 - 2R_1 R_2 \cos \varphi' + (z-z')^2}} \, d\varphi' d\varphi' $$

Mutual inductance for two coaxial, thin, round coil can be defined by substituting (14) to the (7)

$$ M_{C/C} = \frac{\mu_0 N_1 N_2}{2 h_1 h_2} R_1 R_2 \int \frac{\cos \varphi'}{\sqrt{R_1^2 + R_2^2 - 2R_1 R_2 \cos \varphi' + (z-z')^2}} \, d\varphi' d\varphi' $$

This expression can be rewritten by elliptical integrals as:
\[
M_{C/C} = \frac{4}{3} \mu_0 (R_2 R_3)^2 \frac{N_1 N_2}{h_1 h_2} \left[ X(h_{11}) - X(h_{22}) - X(h_{33}) + X(h_{44}) \right]
\]  
(16)

Where function \( X(k) \) is defined as:

\[
X(k) = \frac{1}{k} \left( 1 - \frac{k^2}{2} \right) (K(k) - E(k)) + \frac{3\rho - 4}{2}
\]
\[
E(k) = \frac{3}{2} \rho (1 - k) \Pi \left( \frac{k^2 - 2}{\rho - 2}, k \right)
\]

Where \( K(k) \) is elliptical integral of first kind, \( E(k) \) is elliptical integral of second kind and \( \Pi(n, k) \) is elliptical integral of third kind. Elliptical integrals are explained in next section. Variable \( n \) is known as elliptical characteristic and \( k \) is module of the elliptical integrals [2]. Modules of the elliptical integrals for two thin, round coil are defined as:

\[
k_{11} = \frac{4R_1 R_2}{\sqrt{(R_1 + R_2)^2 + (z_{22} + z_{11})^2}}
\]
\[
k_{22} = \frac{4R_1 R_2}{\sqrt{(R_1 + R_2)^2 + (z_{22} + z_{12})^2}}
\]
\[
k_{33} = \frac{4R_1 R_2}{\sqrt{(R_1 + R_2)^2 + (z_{21} + z_{11})^2}}
\]
\[
k_{44} = \frac{4R_1 R_2}{\sqrt{(R_1 + R_2)^2 + (z_{21} + z_{12})^2}}
\]

Axial distance in (18) is:

\[
z_{22} - z_{11} = d_{11} = d + \frac{h_1}{2} + \frac{h_2}{2}
\]
\[
z_{22} - z_{12} = d_{22} = d - \frac{h_1}{2} + \frac{h_2}{2}
\]
\[
z_{21} - z_{11} = d_{33} = d + \frac{h_1}{2} - \frac{h_2}{2}
\]
\[
z_{21} - z_{12} = d_{44} = d - \frac{h_1}{2} - \frac{h_2}{2}
\]

where \( d \) is distance between middles of the coils. Parameter \( \rho \) \((r, x, i)\) gives relative values of the radius of round coils. Parameter \( \rho \) in this case is:

\[
\rho = \rho(R_1 R_2) = \frac{(R_1 R_2)^2}{2R_1 R_2}
\]

If we can determine expression for self – inductance, we can use expression (16), but distance between middles of the coils is equal zero. In this case radius \( R_1 \) and \( R_2 \) are same. Parameter \( \rho \) is there determined as

\[ R_1 \rightarrow R_2 \Rightarrow \rho \rightarrow 2^+ \]  
(21)

Elliptic integral of third kind is for this parameter \( \rho \)

\[
\lim_{\rho \to 2^+} \Pi \left( \frac{k^2 - 2}{\rho - 2}, k \right) = 0.
\]

Function \( X(k_{22}) \) and \( X(k_{33}) \) from (16) are equal to 1 because:

\[
d_{22} = d_{33} = 0 \Rightarrow k_{22} = k_{33} = 1
\]

\[
\lim_{\rho \to 2^+} \frac{1-k^2}{k^2} (K(k) - E(k)) = 0
\]

\[
\lim_{\rho \to 2^+} \frac{3\rho - 4}{2} E(k) = \frac{3\rho - 4}{2} = 1
\]

For function \( X(k_{11}) \) and \( X(k_{44}) \) we can write

\[
X(k_{11}) = X(k_{44}) = X(k) = \frac{1}{k} \left[ \frac{1-k^2}{k^2} (K(k) - E(k)) + E(k) \right]
\]

Because

\[
|z_{22} - z_{11}| = |z_{21} - z_{12}| = h \Rightarrow
\]

\[
k_{11} = k_{44} = k = \frac{4R_1 R_2}{\sqrt{(R_1 + R_2)^2 + h^2}}
\]

When substituting (22) and (26) to (16), then we can write self - inductance as

\[
L_{C/C} = \frac{8}{3} \mu_0 R_1^2 \frac{X^2}{h^2} \left[ \frac{1-k^2}{k} (K(k) - E(k)) + E(k) \right] - 1
\]

(28)

3. Elliptic integrals

Absolute elliptic integral first kind is defined as:

\[
K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}
\]

(29)

Substitution \( t = \sin \varphi \) we get a normalized form:

\[
K(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2 \sqrt{1-k^2 t^2}}}
\]

(30)

For \( k \in [0,1] \) has this integral real values and for limit values of \( k \) is:

\[
K(0) = \pi/2, \ K(1) = \infty
\]

(31)

Absolute elliptic integral second kind is defined as
E(k) = \int_{0}^{\pi/2} \sqrt{1-k^2 \sin^2 \varphi} \, d\varphi \quad (32)

Substitution \( t = \sin \varphi \) we get a normalized form

\[ E(k) = \int_{0}^{1} \sqrt{1-t^2} \, dt \quad (33) \]

And

\[ E(0) = \pi/2, \quad E(1) \to \infty \quad (34) \]

Absolute elliptic integral of the third kind is defined as:

\[ \Pi(n,k) = \int_{0}^{\pi/2} \frac{1}{\sqrt{1-n \sin^2 \varphi} \sqrt{1-k^2 \sin^2 \varphi}} \, d\varphi \quad (35) \]

Substitution \( t = \sin \varphi \) we get a normalized form

\[ \Pi(k) = \int_{0}^{1} \frac{1}{\sqrt{(1-nt^2)(1-t^2)(1-k^2 t^2)}} \, dt \quad (36) \]

and

\[ \Pi(n \to \infty, k) \to 0, \quad \Pi(n=0, k) = K(k) \]

\[ \Pi(n \to 1, k) \to 0 \]

4. Program for calculation mutual and self inductance

Program for calculation of self - inductance and mutual inductance of two air coils is written in Matlab/Simulink. This can be divided to three parts (fig. 4). First part is where we define main parameters of both coils e.g. radius or height of coil. Main parameters of coils were taken over from existing system for wireless energy transfer. Second part is calculation of self – inductance. At first, the program calculates module the elliptic integrals and then next subsystem calculates self – inductance.

In this subsystem interpreted matlab function blocks are used, where elliptic integrals by function mfun are defined [3] - [5]. With this block, program can calculate absolute elliptic integrals of first, second and third kind. This subsystem calculates value of self – inductance by using (28) and value of the self-inductance of coils is displayed at display block. Third part is similar as second, but this part is for calculation of the mutual inductance of two wire coils. There at first parameter \( \rho \) by using (20) are calculated, axial distances \( d_{11}, d_{22}, d_{33}, d_{44} \) by using (19) and modules of elliptical integrals \( k_{11}, k_{22}, k_{33}, k_{44} \) by using (18). Then in next subsystem mutual inductance by using (16) is calculated and value of the mutual inductance of coils are displayed at display block.

<table>
<thead>
<tr>
<th>distance [mm]</th>
<th>M [uH]</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1.6247</td>
</tr>
<tr>
<td>120</td>
<td>1.1949</td>
</tr>
<tr>
<td>140</td>
<td>0.899</td>
</tr>
<tr>
<td>160</td>
<td>0.6891</td>
</tr>
<tr>
<td>180</td>
<td>0.537</td>
</tr>
</tbody>
</table>

Tab. 1. Calculated mutual inductance in Matlab.

5. Experimental verification on a physical sample

As was already mentioned, the main parameters of the coils were taken over from existing system for wireless energy transfer. The transmitting and receiving coils were designed with helical geometry, whereby coil former was prepared from cardboard paper. In this way, the proper shape and turn’s spacing were secured.

As high frequency generator the evaluation board EPC9003 (fig. 5) has been used [6]. Its main specification is the use of perspective GaN power transistor devices (eGaN EPC2010C) which are suitable for very high frequency operation. This evaluation board also contains necessary equipment for driving these power devices and for sensing required electrical variables in the power circuit.
In Table 2 the main characteristic parameters of evaluation board are listed.

<table>
<thead>
<tr>
<th>Sym.</th>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>VDD</td>
<td>Gate drive Input Supply Range</td>
<td>7-12</td>
<td>V</td>
</tr>
<tr>
<td>VIN</td>
<td>Bus Input Voltage Range</td>
<td>0 - 170</td>
<td>V</td>
</tr>
<tr>
<td>VOUT</td>
<td>Switch Node Output Voltage</td>
<td>0 - 200</td>
<td>V</td>
</tr>
<tr>
<td>IOUT</td>
<td>Switch Node Output Current</td>
<td>&lt;5</td>
<td>A</td>
</tr>
<tr>
<td>VPWM</td>
<td>PWM Logic Input Voltage THR 'High'</td>
<td>3.5 – 6</td>
<td>V</td>
</tr>
<tr>
<td></td>
<td>PWM Logic Input Voltage THR 'Low'</td>
<td>0 – 1.5</td>
<td>V</td>
</tr>
<tr>
<td>min</td>
<td>Min. 'High' State Input Pulse Width</td>
<td>60&lt;</td>
<td>ns</td>
</tr>
<tr>
<td>min</td>
<td>Min. 'Low' State Input Pulse Width</td>
<td>500&lt;</td>
<td>ns</td>
</tr>
</tbody>
</table>

**Tab. 2.** Performance summary of a EPC9003C (TA=25°C)

Table 3 shows characteristic parameters of proposed helical coils for wireless energy transfer. These parameters have been computed based on the target application of wireless system, whose main parameters are parasitic resistance $R$ and quality factor $Q$. Quality factor is derived from other coil’s parameter. Target application of future proposal of this system shall be wireless charging of e-vehicles, thus existing model serves as testing sample in reduced ratio.

<table>
<thead>
<tr>
<th>Param.</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>Diameter of a air coil</td>
<td>185</td>
<td>mm</td>
</tr>
<tr>
<td>l</td>
<td>Coil length</td>
<td>60</td>
<td>mm</td>
</tr>
<tr>
<td>a</td>
<td>Wire diameter</td>
<td>1.5</td>
<td>mm</td>
</tr>
<tr>
<td>N</td>
<td>Number of turns</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>p</td>
<td>Pitch between turns</td>
<td>10</td>
<td>mm</td>
</tr>
<tr>
<td>lW</td>
<td>Physical length of used wire</td>
<td>3486</td>
<td>mm</td>
</tr>
<tr>
<td>L</td>
<td>Inductance</td>
<td>9.34</td>
<td>uH</td>
</tr>
<tr>
<td>R</td>
<td>Effective serial AC resistance</td>
<td>0.062</td>
<td>Ohm</td>
</tr>
<tr>
<td>C</td>
<td>Parasitic capacitance</td>
<td>1020</td>
<td>pF</td>
</tr>
<tr>
<td>Q</td>
<td>Quality factor</td>
<td>174</td>
<td></td>
</tr>
</tbody>
</table>

**Tab. 3.** Computed parameters of proposed coils for wireless power transfer

After coil construction, the main parameters influencing power transfer have been measured (self-inductance, parasitic resistance). These parameters are listed in next table.

<table>
<thead>
<tr>
<th>Param.</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lp</td>
<td>Inductance</td>
<td>9.86</td>
<td>uH</td>
</tr>
<tr>
<td>Ls</td>
<td>Inductance</td>
<td>10.12</td>
<td>uH</td>
</tr>
<tr>
<td>Rp</td>
<td>Serial DC resistance</td>
<td>1.09</td>
<td>Ohm</td>
</tr>
<tr>
<td>Rs</td>
<td>Serial DC resistance</td>
<td>0.8</td>
<td>Ohm</td>
</tr>
</tbody>
</table>

**Tab. 4.** Main electrical parameters of proposed coils

Because proposed wireless system operates on the principle of resonance, the required capacitor with high voltage sustainability and low dielectric absorption was constructed and connected in series with the coil both on transmitter and receiver side.

Fig. 6. Experimental sample of energy transfer.

For the investigation of mutual inductance between transmitter and receiver, it was necessary to measure input current of transmitting coil $I_p$ and voltage on the load $U_Z$. This is due to fact, that mutual inductance can not be measured directly. The formula for the computation is:

$$M = \frac{U_Z}{\omega I_p}$$

Consequently we have determined the relative error between measured and computed values of mutual inductance.

$$\Delta(\%) = \frac{M_{meas} - M_{cal}}{M_{cal}} \cdot 100$$

<table>
<thead>
<tr>
<th>dis [mm]</th>
<th>$U_Z$ [V]</th>
<th>$I_p$ [A]</th>
<th>$M_{meas}$ [uH]</th>
<th>$M_{cal}$ [uH]</th>
<th>$\Delta$ [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>26.6</td>
<td>8.4</td>
<td>1.75362</td>
<td>1.6247</td>
<td>6.54</td>
</tr>
<tr>
<td>120</td>
<td>23.8</td>
<td>10.4</td>
<td>1.2673</td>
<td>1.1949</td>
<td>4.56</td>
</tr>
<tr>
<td>140</td>
<td>19.8</td>
<td>11.6</td>
<td>0.94523</td>
<td>0.899</td>
<td>5.39</td>
</tr>
<tr>
<td>160</td>
<td>15.8</td>
<td>12.4</td>
<td>0.7056</td>
<td>0.6891</td>
<td>3.88</td>
</tr>
<tr>
<td>180</td>
<td>13.2</td>
<td>12.8</td>
<td>0.571</td>
<td>0.537</td>
<td>4.59</td>
</tr>
</tbody>
</table>

**Tab. 5.** Measured parameters on WET system for mutual inductance calculation

**Fig. 5.** The description of a figure is of the style *Description (8pt)*; the figure itself is of the style *Figure.*
6. Conclusion

In this paper, we were forced to evaluate the accuracy of the computation of mutual inductance between two coils, whose might be suited for wireless energy transfer. The algorithm for the computation of mutual inductance was designed based on [1].

The received results show that measured values are in acceptable accordance with computed results, whereby maximal relative error was 7.88%. The main process which probably influenced the accuracy of measurement was existence of frequency and temperature dependent parasitic components, mainly ESR of capacitors (it was series connection of set of capacitors).

Future work will be focused on investigation of parasitic components, which are influencing mutual inductance. Optimization process for their suppression will be done.

Acknowledgements

The authors wish to thank to Slovak grant agency APVV for project no. APVV-0433-12 - Research and development of intelligent system for wireless energy transfer in electromobility application.

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About Authors...

Marek Píri was born in Sahy, Slovakia. He graduated study at University of Zilina (2006). Nowadays study at Ph.D. grade at Department of mechatronics and electronics at University of Zilina. He is interested in the field of power electronics - switch mode power supplies, simulations, design of power supplies.

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