Performance Analysis of Hierarchical Decision Aided 2-Source BPSK H-MAC CSE with Feed-Back Gradient Solver for WPNC Networks

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Abstract—The paper addresses a problem of channel phase estimation in a 2-source Hierarchical MAC channel in a Wireless Physical Layer Coded (WPNC) system with Hierarchical Decode and Forward strategy. We assume a non-pilot based estimator, a pilot based one would require orthogonal pilot signal resources which are not available in a WPNC system. In such a system, the receiving relay node does not have the individual source data available and the only data related estimator aid is a many-to-one network coded data function. We analyze the properties of the estimator metric and the performance of the estimator with a feed-back gradient solver. Particularly we analyze: (1) mean square error (MSE) performance including its relation to Cramer-Rao Lower Bound (CRLB), and (2) we analyze properties of ambiguity modes of the estimator.

I. INTRODUCTION

Wireless Physical Layer Network Coding (WPNC) is a PHY layer concept for communication in dense radio networks with highly interacting signals. We have reached a relatively solid understanding of some fundamental limits, the system design and performance analysis in basic topologies and scenarios [1], [2], [3], [4], [5]. This includes knowledge of achievable and converse rates for specific strategies and scenarios (e.g. Compute & Forward, Noisy Network Coding, Hierarchical Decode and Forward (HDF)), design of Network Coded Modulation (NCM) and related hierarchical demodulation and decoding strategies including their performance analysis. Most of the results assume relatively idealized assumptions related to relative channel parametrization and NCM codebooks used at source nodes. This paper builds on the results of [6] and focuses on the metric properties and performance analysis of the H-MAC channel phase feed-back gradient solver. Particularly we analyze: (1) mean square error (MSE) performance including its relation to Cramer-Rao Lower Bound (CRLB), and (2) we analyze properties of ambiguity modes of the estimator.

II. SYSTEM MODEL

A. 2-source H-MAC Channel

We assume a three-node network with two source nodes (SA, SB) and one relay R. Further we assume a perfect symbol-timing synchronization among all three nodes. The network operates at the H-MAC stage, where the hierarchical target message on the relay is a many-to-one function (Hierarchical Network Code (HNC) map) of both source messages \( b_1 = \chi(b_A, b_B) \). At both source nodes, the messages are encoded by a codebook \( \mathcal{C} = \mathcal{C}(b) \), and symbol-wise mapped on the constellation points \( s_{A,n} = A_n(c_{A,n}), s_{B,n} = A_n(c_{B,n}) \) using a one-dimensional alphabet \( A_n \) of size \( M_n \). The target H-code symbols are denoted as \( c = \mathcal{C}(b) \). We assume an isomorphic layered NCM which implies that \( c_n = \chi(c_{A,n}, c_{B,n}) \). Further we choose a minimal HNC map \( \chi_{c} \), such that given any two elements from \( \{c_n, c_{A,n}, c_{B,n}\} \), we can uniquely determine the third one.

We assume memoryless 2-source H-MAC AWGN channel

\[
x = \mathbf{u}(c_A, c_B) + w = h_A s_A(c_A) + h_B s_B(c_B) + w,
\]

where the fading coefficients are decomposed into magnitude and true phase \( h_A = e^{j\phi_A}, h_B = \eta e^{j\phi_B}, \eta \in \mathbb{R}^+ \) and \( w \) is the AWGN with \( \sigma_w^2 \) variance per dimension. The observed frame is of length \( N \). The SNR will be related w.r.t. SA and denoted as \( \gamma = E\{|s_A|^2|s_B|^2\}/\sigma_w^2 \).

In this paper we consider the special case of a BPSK alphabet \( A_n = \{\pm 1\} \) and binary coded symbols \( c_A, c_B \in \{0, 1\} \) on both source nodes. The natural H-constellation mapper

\[
s(c) = 2c - 1 \text{ is used. In this special case it follows, that the only minimal HNC map is accomplished by the XOR function } c = c_A \oplus c_B, \quad c \in \{0, 1\}.
\]

B. H-MAC Channel Phase Invariance

Since the decoded symbol at R is a many-to-one function, multiple source symbol combinations \( s_A(c_A), s_B(c_B) \) correspond to one H-code symbol \( c \). This phenomenon is called hierarchical dispersion. In our observation model, it demonstrates through the phase ambiguity. For one symbol payload observation part

\[
u(\varphi, c) = \chi_{c}(c_A, c_B) = e^{j\varphi_A} s_A(c_A) + \eta e^{j\varphi_B} s_B(c_B)
\]

it holds that \( u(\varphi, c) = u(\varphi + [(2k_1+1)\pi, (2k_2+1)\pi]) \), where \( k_1, k_2 \in \mathbb{Z} \).

The result shifts in a sign change of both source constellation points which is equivalent to a flip of both source symbols. Since \( \chi_{c}(c_A, c_B) = \chi_{c}(1-c_A, 1-c_B) \), the target symbol is not affected.

This means that from the perspective of the relay we can define a rectangle \( \mathcal{P} = \{[\varphi_A, \varphi_B] : -\pi/2 \leq \varphi_A < \pi/2, -\pi < \varphi_B \leq \pi \} \) such that \( \varphi \in \mathcal{P} \) are unambiguous w.r.t. c. All other solutions are equivalent from the perspective of \( H\)-data aid

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1A prefix ‘H’ denotes hierarchical entities, i.e. many-to-one functions of source node entities (see [1] for details).
CSE. Notice that it does not generally hold for a classical full data $c_A, c_B$ aided estimator.

C. H-MAC Channel Phase Dependence on H-symbol

From the HNC map it follows that changing one source symbol will flip the target symbol. When translated to a change in the channel phase, we get $(k \in \mathbb{Z})$

$$u(\varphi, c) = u(\varphi + (2k + 1)\pi[1, 0]^T, 1 - c), \quad (2)$$

$$u(\varphi, c) = u(\varphi + (2k + 1)\pi[0, 1]^T, 1 - c). \quad (3)$$

III. H-MAC Phase Estimator

A. Estimator H-data Aided Metric [6]

Having a memoryless, AWGN channel and symbol-wise constellation mappers, the observation likelihood can be factorized as $p_N(x|\varphi, c_A, c_B) = \prod_{n=0}^N p(x|\varphi, c_A, n, c_B, n)$, where we assume the coefficient $\eta$ as well as the hierarchical H-code symbols $c$ to be known at $R$. The final metric is obtained by marginalization over the H-dispersion. Since the HNC map is minimal, we get

$$p(x|\varphi, c) = \frac{1}{P(c)} \sum_{s_A} p(x|\varphi, s_A, s_B(s_A, c)) p(s_A) p(s_B). \quad (4)$$

Consider our special case and IID source symbols, after some manipulations we obtain the logarithm of the scaled (dropping scaling independent on $\varphi_A, \varphi_B$) $p(x|\varphi_A, \varphi_B, c)$ as

$$\rho = -\frac{2}{\sigma_w^2} \eta s(c) \cos(\varphi_A - \varphi_B)$$

$$+ \ln \cosh \left( \frac{2}{\sigma_w^2} \left[ \Re \left[ x e^{-j \varphi_A} \right] + s(c) \Re \left[ x e^{-j \varphi_B} \right] \right] \right), \quad (5)$$

where $s(c) = s_A s_B = 1 - 2c$ follows from the XOR HNC map. The ML estimator is then given by

$$\hat{\varphi} = [\hat{\varphi}_A, \hat{\varphi}_B] = \arg \max_{\varphi} \left\{ \rho_N(\varphi, c) = \sum_{n=1}^N \rho(\varphi, c_n) \right\}. \quad (6)$$

B. Location of the Metric stationary points

In this section we analyze the locations of stationary points of the metric given by

$$\rho_N(\varphi) = \sum_{n=1}^N -\frac{2}{\sigma_w^2} \eta s(c_n) \cos(y_A - y_B)$$

$$\sum_{n=1}^N \ln \cosh \left( \frac{2}{\sigma_w^2} \left[ \Re \left[ x_n e^{-j \varphi_A} \right] + s(c_n) \Re \left[ x_n e^{-j \varphi_B} \right] \right] \right). \quad (7)$$

Let us assume a very long frame $N \to \infty$ and equiprobable target symbols $Pr(c_n = 1) = Pr(c_n = 0)$. Those assumptions are reasonable, since frame lengths of today used LDPC codes are in the order of 64800, and a minimal HNC map has equiprobable $c$. Using those assumptions we get

$$\frac{2}{\sigma_w^2} \cos(\varphi_A - \varphi_B) \sum_{n=1}^N s(c_n) \to 0, \quad (8)$$

while the second term of (7) is a monotonically increasing function of $N$ since $\ln(cosh(x)) \geq 0$, $\forall x$. Further if we assume operation in the high-SNR regime, $\frac{\sigma_w^2}{\sigma_n^2} \gg 1$, we can use the approximation $\ln(cosh(x)) \approx \ln(0.5) + |x|$ for $x \gg 1$.

After dropping additive and multiplicative constants $\ln \left( \frac{1}{2} \right)$, we obtain

$$\rho''_N(\varphi) \approx \sum_{n=1}^N \left[ \Re \left[ x_n e^{-j \varphi_A} \right] + s(c_n) \Re \left[ x_n e^{-j \varphi_B} \right] \right]. \quad (9)$$

Since this is an off-line analysis we can substitute $x_n$ according to (1) with the true phase vector $\phi = [\phi_A, \phi_B]$, after neglecting the noise completely we obtain

$$\rho''_N(\varphi) \approx \sum_{n=1}^N \left[ \Re \left[ x_n e^{-j \phi_A} \right] + s(c_n) \Re \left[ x_n e^{-j \phi_B} \right] \right]. \quad (10)$$

Since $s_A, s_B \in \{\pm 1\}$, each n-th term in (10) can be multiplied by $s_A, n$ not changing the overall sum value. When applying this we get the final approximate expression as

$$\rho''_N(\varphi) \approx \sum_{n=1}^N \left[ \cos(\phi_A - \phi_B) + s(c_n) \cos(\phi_A - \phi_B) \right. \left. + \eta s(c_n) \cos(\phi_B - \phi_A) + s(c_n) \cos(\phi_B - \phi_B) \right]. \quad (11)$$

We notice, that $c_n$ is the only part that depends on the summation index. That is as expected, realizing that the relay metric is marginalized over the hierarchical dispersion. Using the assumption of $c_n$ having equiprobable values and $N$ being large, the sum can be reordered and split in half according to the value of $s(c_n) \in \{\pm 1\}$. Dropping the multiplicative $\frac{N}{2}$ we get

$$\rho''(\varphi) = \frac{2}{N} \rho''_N(\varphi) = \rho''(\varphi; 1) + \rho''(\varphi; -1), \quad (12)$$
where

\[
\rho''(\varphi, s) = [\cos(\varphi_A - \varphi_B) + \eta s \cos(\varphi_A - \varphi_B)] + \eta s \cos(\varphi_B - \varphi_A) + \eta^2 \cos(\varphi_B - \varphi_B)].
\tag{13}
\]

Now we analyze the arguments of extrema separately for each of the four sign combinations of the moduli arguments of \(\rho''(\varphi, 1), \rho''(\varphi, -1)\) in (12).

1) \([+ , +] \): With both moduli arguments of \(\rho''(\varphi, \pm 1)\) positive, we can remove the moduli and simplify to \(\rho''_+(\varphi) = 2\cos(\varphi_A - \varphi_B) + 2\eta^2 \cos(\varphi_B - \varphi_B)\), performing the first and second derivative w.r.t. \(\varphi\) it yields

\[
\Delta_\varphi = [2\sin(\varphi_A - \varphi_B), 2\eta^2 \sin(\varphi_B - \varphi_B)]^T,
\]

\[
H_\varphi = \text{diag}(-2\cos(\varphi_A - \varphi_B), -2\eta^2 \cos(\varphi_B - \varphi_B)),
\]

where \(\Delta\) denotes the gradient and \(H\) the hessian matrix. Clearly we see that both \(\varphi_0 = [\varphi_A, \varphi_B]^T, \varphi_1 = [\varphi_A, \varphi_B + \pi]^T\) zero the gradient. From the periodicity of the metric we get an infinite number of other stationary points, but in terms of Section II-B they are equivalent to \(\varphi_0\) and \(\varphi_1\). Since \(H_\varphi|_{\varphi_1}\) is negative definite and \(H_\varphi|_{\varphi_1}\) is indefinite, we get a maximum at \(\varphi_0\) and a saddle point at \(\varphi_1\). Now we need to check if the assumption of positive moduli arguments is met. After a substitution into (12) we get the following conditions

\[
\eta^2 \pm 2\eta \cos(\varphi_A - \varphi_B) + 1 \geq 0 \text{ for } \varphi_0,
\]

\[
1 - \eta^2 \geq 0 \text{ for } \varphi_1.
\]

(15) (16)

Since \(\eta^2 \pm 2\eta \cos(\varphi_A - \varphi_B) + 1 \geq \eta^2 - 2\eta + 1 = (\eta - 1)^2 \geq 0\) and \(0 < \eta \leq 1\), both conditions fulfilled. This proves that the metric will always have a maximum for the true phase vector.

2) \([+, -] \): We will proceed similarly as in Section III-B1:

\[
\rho''(-\varphi) = 2\eta \cos(\varphi_B - \varphi_A) + \cos(\varphi_A - \varphi_B),
\]

\[
\Delta_\varphi = [2\eta \sin(\varphi_B - \varphi_A), 2\eta \sin(\varphi_A - \varphi_B)]^T
\]

\[
H_\varphi = \text{diag}(-2\eta \cos(\varphi_B - \varphi_A), -2\eta \cos(\varphi_A - \varphi_B)).
\]

(17) (18) (19)

Here the stationary points are given as \(\varphi_0 = [\varphi_B, \varphi_A]^T, \varphi_1 = [\varphi_B, \varphi_A + \pi]^T\), again a maximum and a saddle point respectively. Next we evaluate the sign conditions:

\[
(\eta^2 + 1) \cos(\varphi_A - \varphi_B) \pm 2\eta  \geq 0 \text{ for } \varphi_0,
\]

\[
(1 - \eta^2) \cos(\varphi_A - \varphi_B) = 0 \text{ for } \varphi_1.
\]

(20) (21)

In this case (20) is not trivially fulfilled. The shaded area in (Fig. 2) contains all points where there is a false maximum of the metric.

Remark 1. We may notice, that for the special case when \(\eta = 1\) the observation model yields the exactly same result for \([\varphi_A, \varphi_B]\) with source symbols \(c_A \neq c_B\) and \([\varphi_B, \varphi_A]\) with flipped source symbols \(1 - c_A, 1 - c_B\) for \(c_A = c_B\) the source symbols need not to be flipped). From the relay point of view view these two cases are equivalent since the target symbol \(c\) stays the same.

\[
u([\varphi_A, \varphi_B]^T, \chi c(c_A, c_B)) = u([\varphi_B, \varphi_A]^T, \chi c(1 - c_A, 1 - c_B))
\]

This corresponds to the existence of a second maximum independently on the value of \(\cos(\varphi_A - \varphi_B)\), see (Fig. 2).

3) \([- , -], [- , +] \): Here we would proceed the same way as in Sections III-B1 and III-B2. Since we would not obtain any additional maxima or saddle points, we do not investigate those cases further.

C. Value of Metric Maxima

In this section we are interested in the absolute values of the metric maxima and relations among them. For the derivation we use the same approximation (12) and fixing the absolute error introduced by dropping constants

\[
\rho_N(\hat{\varphi}) \approx N \ln(1/2) + \frac{N}{2} \frac{\eta}{\sigma_w^2} (\rho''(\varphi, 1) + \rho''(\varphi, -1)).
\tag{22}
\]

Evaluating the metric at the true phase vector we get

\[
\rho_N(\varphi)|_{\varphi = [\varphi_A, \varphi_B]} \approx N \ln \left(\frac{1}{2}\right) + \frac{2N}{\sigma_w^2} (1 + \eta^2).
\tag{23}
\]

At the false phase vector the metric evaluates to

\[
\rho_N(\varphi)|_{\varphi = [\varphi_B, \varphi_A]} \approx N \ln \left(\frac{1}{2}\right) + \frac{2N}{\sigma_w^2} (2\eta_B).
\tag{24}
\]

For the difference \(\rho_N(\varphi)|_{\varphi = [\varphi_A, \varphi_B]} - \rho_N(\varphi)|_{\varphi = [\varphi_B, \varphi_A]}\) we can write

\[
\frac{2N}{\sigma_w^2} (1 + \eta^2 - 2\eta) = \frac{2N}{\sigma_w^2} (1 - \eta)^2 \geq 0.
\tag{25}
\]

We can conclude, that for \(\eta \neq 1\) the metric attains a global maximum at the true phase vector and for fulfilled conditions (20) a local maximum at the flipped phase vector. In the singular case when \(\eta = 1\) there are two equally valued local maxima, however in this situation the true phase vector yields the same observation as the false one (see Remark 1).

IV. FEED-BACK GRADIENT SOLVER

A. CANONIC FORM OF THE SOLVER

The extreme of \(\rho_N\) can be found using a feed-back iterative solver with additive updates given the estimate updates

\[
\hat{\varphi}(i + 1) = \hat{\varphi}(i) + K_1 \mu_N,
\tag{26}
\]

where \(K_1\) is the properly chosen step-size and \(\mu_N = \nabla_{\varphi} \rho_N(\varphi(i), c)\) is the gradient with symbol-wise components.
\( \mu_{N,A} = \frac{\partial P}{\partial \varphi_A} \) and \( \mu_{N,B} = \frac{\partial P}{\partial \varphi_B} \). After carrying out the partial derivatives, we get (see details in [6])

\[
\mu_{N,A} = \sum_{n=1}^{N} \frac{2}{\sigma_v^2} \eta_s(c_n) \sin(\varphi_A - \varphi_B) + \frac{2}{\sigma_v^2} \eta_s(c_n) x_n e^{-j\varphi_A}
\]

\[
\times \tanh \left( \frac{2}{\sigma_v^2} \left( \Re \left[x_n e^{-j\varphi_A} \right] + \eta_s(c_n) \Re \left[x_n e^{-j\varphi_B} \right] \right) \right)
\]

and similarly for \( \mu_{N,B} \).

B. Proposed Estimator Algorithm

In this section we propose an iterative hierarchical data decision aided estimator algorithm. We assume the channel phase to be constant over the whole frame. Let us define the decision aided estimator algorithm. We assume the channel is the observation frame length.

The algorithm is carried out as follows. First \( \hat{\varphi}(0) \) is set to an initial value and then the loop (26) is started. In each iteration the condition \( \epsilon > \mu_{N}^2 K_1 K_1 \mu_{N} \) is evaluated. When true in the \( i \)-th iteration, we evaluate \( \rho_N \) at

\[
\begin{bmatrix}
\varphi_A(i) \\
\varphi_B(i)
\end{bmatrix}
\]

and find the maximum. If we get the maximum at \( [\hat{\varphi}_A(i), \hat{\varphi}_B(i)]^T \), we take it as the final estimate. If not, we take the corresponding vector and use it as the starting point and restart the algorithm. To get an unambiguous result, we shift the final estimate according to Section II-B such that it lies in \( \mathbb{P} \).

V. CRLB Analysis of the CSE

For each frame, the phase vector \( \varphi \) is assumed constant and our estimator is unbiased. Thus we can take the CRLB as the theoretical lower limit for the estimate variance over multiple realizations. First we compute the Fisher information matrix

\[
\bf{J}_{k,i}(\varphi) = -E \left[ \frac{\partial^2 \ln(p(x|\varphi))}{\partial \varphi_k \partial \varphi_i} \right]
\]

The expectation is evaluated numerically using Monte Carlo simulation. The individual estimate component variances are then compared with the corresponding diagonal elements of the inverse information matrix, while it must hold that \( \text{var} \left[ \hat{\varphi}_k \right] \geq \frac{1}{N} \left[ \bf{J}^{-1}(\varphi) \right]_{k,k} \) where \( N \) is the observation frame length.

VI. Numerical Results

In this section we compare a simulation based MSE performance of the algorithm proposed in Section IV-B with the CRLB. We use a frame length of \( 10^4 \) symbols and a randomly taken initialization phase vector for each frame. The loop gain and threshold were experimentally chosen \( K = 1, \epsilon = 0.01. \)

In (Fig. 3) we plot the variance of the estimates for different values of SNR together with the theoretical lower bound to verify that the estimator performance approaches the limit.

VII. Discussion and Conclusions

In the paper, we have analyzed the system model of the 2-source network from the point of view of a phase H-MAC CSE. We showed that there are two phase ambiguity types. One resulting from the constellation symmetry and having effect on the hierarchical target symbol and the other resulting from the hierarchical dispersion which does not affect the target symbol.

In Section III we analyzed the H-MAC CSE metric stationary points with the conclusion that under certain conditions there exists a second local (false) maximum. We showed that there is a direct relation between individual stationary points and that it is possible to identify the global maximum at the receiver based on the corresponding value of the metric.

Based on those observations we proposed a feed-back gradient based estimator algorithm using a full-frame observation. From (Fig. 3) we see that the estimator performance approaches the CRLB.

REFERENCES


