Achievable Rates for HDF WPNC Strategy with Hierarchical Bit-Wise Network Coding Maps for Higher-Order Constellations in H-MAC Channel with Relative Fading

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Abstract—The paper addresses Wireless Physical Layer Network Coding (WPNC) with Hierarchical Decode and Forward (HDF) strategy. We analyze achievable hierarchical rates in one stage hierarchical MAC (H-MAC) channel for higher order component constellations with bit-wise Hierarchical Network Code (HNC) maps. This is motivated by a possible application of state-of-the-art binary (e.g. LDPC) codes over higher order constellations. We show that the bit-mapped binary codes do not have the same achievable rates as direct higher-order codes. On top of this, the individual bits in the HNC map might provide very uneven performance and it strongly depends on the combination of component alphabets. The results are supported by a validation with practical LDPC codes.

I. INTRODUCTION

A. Background, and Related Work

Wireless Physical Layer Network Coding (WPNC) gained some maturity in recent years. Various aspects covering a Network Coded Modulation (NCM) and hierarchical decoding design are investigated in many forms and under various system assumptions (e.g. [1], [2], [3], [4]). Most of the results assume relatively idealized assumptions related to relative channel parametrization and NCM codebooks used at source nodes.

This paper focuses on a practical aspect of using standard state-of-the-art binary codes as NCM codebooks in the WPNC system with higher order constellations and parametrized H-MAC channel. These scenarios lead to bit-wise mapped Hierarchical Network Code (HNC) maps.

B. Goals of this Paper and Contributions

The motivation and goals of the paper can be outlined in the following points.

1) Practical state-or-art binary component codebooks imply NCM with bit-wise HNC maps.
2) We focus on H-MAC stage with Hierarchical Decode and Forward (HDF) strategy. The H-MAC stage is a main performance bottleneck of such WPNC system.

The paper provides the following contributions and results.

1) We show that the bit-wise mapped HNC maps imply some significant relative parameterization related performance phenomena in comparison with direct higher-order HNC maps.
2) The performance is evaluated in terms of hierarchical mutual information rates evaluated for various bit-wise HNC mapped higher order constellations and for various signal-to-noise ratio and relative channel parameterization. The results are supported by a validation with practical LDPC codes.

II. SYSTEM MODEL

A. Component Messages and Codes

We assume a 2-component (SA, SB) H-MAC topology (Fig. 1). Component messages are binary vectors $b_1, b_2 \in \mathbb{F}_2^N$. The relay uses the HDF strategy to decode the H-message $b \in \mathbb{F}_2^N$ with a minimum cardinality bit-wise HNC map $b_i = \chi(b_{A,i}, b_{B,i}), i \in \{0 : N_0 - 1\}, b_{A,i}, b_{B,i} \in \mathbb{F}_2^2$. The prefix “H-” is used to denote hierarchical entity. We focus on the performance of H-decoding on one relay and assume that the end-to-end solvability is fulfilled by a proper choice of HNC maps and corresponding information measures on all involved relays and paths to the final destination. The component sources are assumed to be synchronized at the symbol timing level.

The component messages are encoded by a common $(2^{N_R},N)$ binary codebook $\mathcal{C}$ on GF $\mathbb{F}_2$, $c_A = \mathcal{C}(b_A)$, $c_B = \mathcal{C}(b_B)$, $c_A, c_B \in \mathbb{F}_2^N$. The component code rate is $R = N_0/N$. A ‘prime’ notation $(\cdot)'$ is used for symbols from binary GF (additional details will come later).

In a standard form of using binary code with higher-order constellations, the channel symbols transmitted by the component nodes are bit-wise grouped into $m$ dimensional vectors $c_{A,n}', c_{B,n}'$ which are in turn used to make $M$-ary channel symbols, $M = 2^m$, $s_{A,n} = \phi_A(c_{A,n}')$, $s_{B,n} = \phi_B(c_{B,n}')$. We generally allow different constellation mappers $\phi_A, \phi_B$ on both sources. Since the correspondence between extended GF and bit-wise symbols is one-to-one, we can uniquely find a bit-mapped form of the HNC map for $i \in \{0, \ldots, m-1\}$, $\chi_{i,j}''(c)$.
Alternatively, we may multiplex source message into multiple streams and each encode by a separate encoder (potentially with different rates). The resulting binary streams are then applied to individual bits of $N$, dimensional vectors $\mathbf{c}_A, \mathbf{c}_B$. In both cases, the binary code is linear $\mathbf{c}_A = \mathbf{G}_A^\prime \mathbf{b}_A$, and similar notation for others.

B. Bit-wise Linear HNC Maps on Extended GF

In a simplest case, the linear HNC map and the code itself use the same size of the alphabet. In a special case, when this is an extended GF, $M = M'^m$, we might however use a mix of the operations on $F_M$ and $\mathbb{C}^m_{M'}$, where the latter is a vector space of $m$-tuples on $F_M'$. Particularly attractive seems to be a combination of binary code $M' = 2$ with $M'^m$ sized constellations.

Assume that $M = M'^m$ where $M'$ is prime. All symbols, codes, coefficients with elements from $F_M'$ with be denoted by $(\cdot)'$, e.g. $c_j'$, and the operations on $F_M'$ or the associated vector space (applied element-wise) $\mathbb{C}^m_{M'}$ will be denoted by $\oplus, \otimes$. The symbols and operations on $F_M$ will keep the standard notation for addition and multiplication. The vectors from the vector space on $F_M'$ are segmented into $m$-tuples. We assume that the size of the vector is an integer multiple of $m$. We denote $m$-tuple of $F_M'$ symbols by $\mathbf{c}',$ and vector of $m$-tuples as $\mathbf{c}' = [\ldots, c_1', \ldots]^T,$ and similarly for other variables. The “conversion” function between extended GF and its $m$-tuple representation is $f : \mathbb{C}^m_{M'} \mapsto F_M'$, and its form combining the whole vector from $m$-tuples is $F : F^m_{M'} \mapsto \mathbb{C}^m_{M'}$.

A bit-wise linear HNC map is defined as a map with $F_M'$ coefficients, where the linear HNC operation is applied on $F_M'$ and the result is mapped on the $F_M$ using the $f(\cdot)$ function. The HNC map for code is defined as

$$\mathbf{c} = f(\mathbf{c}') = f(\mathbf{a}_A \otimes \mathbf{G'} \otimes \mathbf{b}_A + \mathbf{a}_B \otimes \mathbf{G'} \otimes \mathbf{b}_B') \quad (1)$$

where $\mathbf{a}_A, \mathbf{a}_B \in F_M'$. This clearly limits the range of the coefficients but preserves the linearity. For example, if $M' = 2$ then the only non-singular choice is $\mathbf{a}_A = \mathbf{a}_B = 1$ (i.e. bit-wise XOR) regardless of the size $M$. Notice that in our particular setup, we assumed binary messages, and thus $\mathbf{b} = \mathbf{b}'$.

The linear operation on $F_M'$ leads to the desired result

$$\mathbf{c} = f(\mathbf{G'} \otimes (\mathbf{a}_A \otimes \mathbf{b}_A' + \mathbf{a}_B \otimes \mathbf{G'} \otimes \mathbf{b}_B'))$$

$$= f(\mathbf{G'} \otimes \mathbf{b}') \quad (2)$$

where H-message HNC map is $\mathbf{b}' = \mathbf{a}_A \otimes \mathbf{b}_A' \otimes \mathbf{a}_B \otimes \mathbf{b}_B'$ and $\mathbf{H}$-code at the level of $F_M'$ GF is $\mathbf{c}' = \mathbf{G'} \otimes \mathbf{b}'$, and the $F_M'$ H-codebook is regular [5]. The H-codebook on $F_M$ is then obtained by $m$-tuple grouping $\mathbf{c} = f(\mathbf{c}')$. If the $F_M'$ code has uniform and IID symbols then also the $m$-tuple groups will be uniform and IID on $F_M$. Thus the regularity condition is fulfilled also for $F_M$ H-codebook.

C. Linear AWGN H-MAC Channel

The channel model is a flat-multipath linear AWGN channel with relative fading coefficient $h = [h] \exp(j \gamma)$,

$$x_n = (s_{A,n} + h s_{B,n}) + w_n \quad (3)$$

where $w_n$ is complex-valued AWGN with $\sigma_w^2$ variance per dimension. We assume a unity common fading coefficient.

Since the common fading cannot influence the performance related to hierarchical entities, it is dropped for a clarity of presentation. Hierarchical channel combined symbol is

$$u_n = s_{A,n} + h s_{B,n} \quad (4)$$

and the H-alphabet is $A_H = \{u_n\}, u_n \in A_H$.

III. HIERARCHICAL BIT-WISE ACHIEVABLE RATES

Here, we evaluate the hierarchical mutual information in a variety of forms — higher order symbol-wise, and bit-wise with different mappings. In the case of isomorphic regular layered NCM, these rates are achievable [5]. We use standard information-theoretic notation, upper and lower case letters denote random variable and its particular value respectively.

We use tilde notation to denote the set of all component source variables, e.g. $\tilde{c} = \{c_A, c_B\}$, and similarly for others. All mutual information evaluations and all rates are single-letter entities. In order to simplify the notation, we can drop the symbol sequential index $n$. The ‘primed’ entities correspond to bit-wise variables and we will use $c'$ to denote one $m$-tuple of bit-wise elements.

A. Symbol-wise Hierarchical Mutual Information

The symbol-wise hierarchical mutual information $I(C;X)$ is evaluated by a standard procedure [5], [4]

$$I_C = I(C;X) = \mathcal{H}[X] - \mathcal{H}[X|C].$$

The evaluation requires the corresponding PDFs

$$p(x) = \sum_{\tilde{c}} p(x|\tilde{c}) p(\tilde{c})$$

$$= \sum_{\tilde{c}} p(x|\tilde{c}) p(\tilde{c})$$

$$= \frac{1}{M^2} \sum_{\tilde{c}} p_w(x - u(\tilde{c})),$$

and

$$p(x|c) = \frac{1}{M} \sum_{\tilde{c} : X(\tilde{c}) = c} p_w(x - u(\tilde{c})),$$

where

$$p_w(w) = \frac{1}{\pi \sigma_w^2} \exp(-\frac{1}{\sigma_w^2} ||w||^2).$$

The entropies are evaluated by Monte-Carlo integration [5], [4].
B. Bit-wise Hierarchical Mutual Information

The bit-wise hierarchical mutual information for individual i-th bit of the m-tuple is

\[ I_i = I(C_i';X) = \mathcal{H}[X] - \mathcal{H}[X|C_i'] \] (9)

In order to evaluate this single-letter one-bit mutual information, we need marginalized PDF \( p(x|\epsilon') \). The i-th bit \( \epsilon'_i \) is a part of m-tuple \( \epsilon' \), and we assume that all bits in \( \epsilon' \) are independent and have a uniform distribution. This corresponds to using idealized binary codebooks. The marginalized PDF is \((M' = 2, M = M^{m})\)

\[
p(x|\epsilon'_i) = \frac{1}{p(\epsilon'_i)} \sum_{\epsilon'_i' : \mathcal{G}_\epsilon' \epsilon'_i' = \epsilon'_i} p(x|\epsilon'_i') p(\epsilon'_i')
= \frac{M'}{M^2} \sum_{\epsilon'_i' : \mathcal{G}_\epsilon' \epsilon'_i' = \epsilon'_i} p(x|\epsilon'_i')
= \frac{M'}{M^2} \sum_{\epsilon'_i' : \mathcal{G}_\epsilon' \epsilon'_i' = \epsilon'_i} p_\epsilon(x - u(\epsilon'_i'))
\]

where we used the fact, that there is a one-to-one mapping between \( \epsilon \) and its bit-mapped form \( \epsilon' \).

C. Relation of Symbol-wise and Bit-wise Hierarchical Mutual Information

It is instructive to investigate the relationship between the bit and the symbol-wise mutual information. For a simplicity, we focus on a special case \( m = 2 \). A one-to-one symbol to bit m-tuple directly implies

\[ I_\epsilon = I(C;X) = I(C_0', C_1';X). \] (11)

An application of the chain rule gives

\[
I(C_0', C_1';X) = I(C_0';X) + I(C_1';X|C_0'), \]
\[ I(C_0', C_1';X) = I(C_1';X) + I(C_0';X|C_1'). \] (13)

This clearly shows that the symbol-wise mutual information \( I(C;X) = I(C_0', C_1';X) \) is simply not a sum of bit-wise mutual information over all bits, which we define as

\[ I_{\epsilon_01} = I_{\epsilon_0} + I_{\epsilon_1} = I(C_0';X) + I(C_1';X). \] (14)

This value does not have any general relationship to \( I_\epsilon \). \( I_\epsilon \leq I_{\epsilon_01} \), since the conditioning of mutual information can both increase or decrease its value.

D. Achievable Rates for Binary Codes

Values of bit-wise mutual information \( I_i \) and \( I_{\epsilon_i} \) might quite substantially differ due to the relative fading in the H-MAC channel. Individual bits may be in a very different situation from that perspective. From the perspective of given individual bit, we can extend the arguments given in [5] to conjecture, that the rate \( R_i = I_i \) is achievable. However, the binary code can be applied as one common code stretched over all bits, or we can apply different codes (from multiplexed data) over individual bits.

In the case of a common code, the individual bits in m-tuple provide different achievable rates. It can be viewed as a communication channel with two random and equally probable states \( \theta \). Using the idea from [6] (Sec. 3.2), the capacity of the receiver-only channel side information channel with IID channel (per-symbol) channel states and the transmitted signal with channel state independent distribution is given by the expectation over the channel states \( E_\theta[C(\theta)] \). The codebook for such a system is constructed for the mean rate \( E_\theta[C(\theta)] \).

For a pair of bits, it practically means that the rate

\[ I_{\epsilon_01} = I_{\epsilon_0} + I_{\epsilon_1} \] (15)

is achievable. The problem of this approach is in the fact that it applies to capacities and perfect idealized codebook. Practical codes might be generally quite sensitive to this phenomenon. However, practical experiments with selected LDPC codes (see Section V) show that the LDPC code performs quite close to the pair-wise sum (or to an average, if it is related to a single bit) of the supported bit-wise rates.

In the case of two independent component codes applied to individual bits, we have a full control of the code rate vs. values \( I_{\epsilon_0} \) and \( I_{\epsilon_1} \) that can be directly used for any practical codes. Again, the overall rate for a pair of bits is

\[ I_{\epsilon_01} = I_{\epsilon_0} + I_{\epsilon_1} \] (16)

however this one can be explicitly mapped to rates of practical codes. This approach requires channel state information on the transmit side.

Notice, that the higher order alphabet \( M \)-ary code would have achievable rate given by \( I_\epsilon \).

IV. Numerical results

The binary mapped hierarchical achievable rates \( I_{\epsilon_0}, I_{\epsilon_1}, I_{\epsilon_01} \) and \( M \)-ary symbol code rates \( I_\epsilon \) were numerically evaluated for various constellation mappers \( \alpha_4, \alpha_6 \). We first plot the minimum, the maximum and the average values over the channel relative fading phase as a function of signal-to-noise ratio. Then we also plot a relative degradation caused by the relative channel fading phase for selected signal-to-noise ratios. Figure 2 shows the results for 4PSK with natural mapping \( \{00,01,10,11\} \mapsto \{1,j,-1,-j\} \) in both sources and Figure 3 for source SA with 4PSK and source SB with 4PSK-cross mapping \( \{00,01,10,11\} \mapsto \{1,-1,j,-j\} \).

V. Practical Validation with LDPC Codes

In this section we present simulation results for the H-MAC stage, using practical binary LDPC codes for the channel coding. Namely the well-known DVB-S2 LDPC codes with available code rates \( R \in \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10} \} \) were used. From the numerical results it follows, that for certain relative parameterizations, the individual bits used in bit-wise hierarchical map show different performances as was predicted in the achievable rate analysis in previous sections. The main objective is to find practical achievable code rates (defined as rates for which the error rate is sufficiently low) in various bit-wise hierarchical mapping scenarios.
For this simulation, we used the 4PSK modulation with natural mapping and different relative parametrization $h = 1 \exp(j \psi)$. First both component messages $b_A$ and $b_B$ are encoded by a binary LDPC code. Then transferred over an H-MAC AWGN channel and hierarchically decoded at the relay using true [5] hierarchical decoding metric and an iterative decoding process with maximum of 30 iterations.

The following three scenarios were simulated:

1) Coded binary component messages $b_A', b_B'$ are directly mapped onto 4-ary channel symbols, only using bit#0 (MSB) of each symbol. Bit#1 is taking a random value. This scenario isolates the behavior of bit#0.

$$\mathcal{A}_B(\cdot) = \mathcal{A}_B(\cdot) : 0 \mapsto 00 \text{ or } 01; 1 \mapsto 11 \text{ or } 01$$

2) Coded binary component messages $b_A', b_B'$ are directly mapped onto 4-ary channel symbols, only using bit#1 (LSB) of each symbol. Bit#0 is taking a random value. This scenario isolates the behavior of bit#1.

$$\mathcal{A}_B(\cdot) = \mathcal{A}_B(\cdot) : 0 \mapsto 00 \text{ or } 01; 1 \mapsto 10 \text{ or } 11$$

3) Coded binary component messages $c_A', c_B'$ are bit-wise grouped into 2 dimensional vectors $c_A', c_B'$ and then they are mapped onto 4-ary channel symbols.

$$\mathcal{A}_B(\cdot) = \mathcal{A}_B(\cdot) : 00 \mapsto 00, 01 \mapsto 01, 10 \mapsto 10, 11 \mapsto 11$$

Graphs in Figures 4, 5, 6 7 (which differ by the relative fading value $h$) show which code rates are practically achievable at each individual SNR, to assure a bit error rate $\leq 10^{-3}$. We can see the performance difference between scenario 1 and scenario 2, which directly corresponds with the strong and weak bit-wise performance according to the relative parametrization $h$. Despite the highly limited granularity of the available code rates, we can conclude, that the bit error rate of scenario 3 lies in-between of those of scenario 1 and 2. This means, that the
Figure 4. 2-component H-MAC stage with LDPC coding. Achievable rates at BER $\leq 10^{-5}$ for natural 4PSK mapping and $\psi = 30$.

VI. DISCUSSION AND CONCLUSIONS

Using state-of-the-art binary (e.g., LDPC) codes over higher order constellations in WPNC system is a highly desirable goal. However, the numerical results show that the bit-mapped binary codes do not achieve the same achievable rates as direct higher-order codes directly applied on higher-order constellation. This is demonstrated by comparing the achievable rates for higher-order alphabet codes and bit-wise mapped binary codes (black and green lines in Figures 2a, 3a). It generally holds for all minimum, maximum, and mean values, however the amount of the performance drop differs. It seems to be the most visible on the minimum values over all channel phases.

We also observe that the bit-wise mapped HNC maps imply some significant relative parameterization related performance phenomena in comparison with direct higher-order HNC maps. Figures 2b, 3b show that, depending on the component constellations, the individual hierarchical bits can undergo quite different performance drop under relative phase fading (compare red and blue lines). For 4PSK and 4PSK case, we see that one of the bits is clearly much more sensitive to the relative phase fading. This phenomenon however disappears for a combination of 4PSK and 4PSK-cross mappings which has nicely even bit-wise performance for both bits.

We see that the application of binary codes over higher order NCM component alphabet in HDF WPNC system might present a significant performance problem in comparison with the application of direct higher order alphabet codes. Also, the individual bits in the HNC map might provide very uneven performance. This phenomenon strongly depends on the combination of NCM component alphabets.

REFERENCES