Systematic Design of Hierarchical Network Code Mapper for Butterfly Network Relaying

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Abstract—A majority of the research results on Wireless Network Coding (WNC) considers only a basic 2-Way Relay Channel (2-WRC) system scenario, where the Complementary Side Information (C-SI) is naturally available at both destinations. Although an extension of the WNC principles from the 2-WRC system to the butterfly network could seem relatively straightforward, several new unconventional phenomena of the WNC arise (and need to be considered). First of all, the unreliable transmission of the C-SI can be overcome by an increased cardinality of the relay output. This opens a question how a suitable Hierarchical Network Code (HNC) mapper could be designed for a given quality of the C-SI link (and source alphabet cardinality). In this paper we focus on this problem. We introduce a systematic approach to the design of a set of HNC mappers, respecting the amount of available C-SI. Observed capacity gains of this (extended cardinality) relaying pave the way for an adaptive butterfly network, where the achievable throughput can be maximized by a suitable choice of the HNC map at the relay.

I. INTRODUCTION

A. Background and Related Work

The principles of Wireless Network Coding (WNC) were built up on a cornerstone of the traditional Network Coding (NC) paradigm [1], utilizing a specific nature of the wireless channels (e.g., inherent broadcast). A plenitude of research results has undoubtedly demonstrated the potential performance gains of the WNC-based wireless systems. However, a majority of the research results on WNC considers only a basic 2-Way Relay Channel (2-WRC) system scenario (see e.g. [2], [3], [4], [5], [6], [7], [8], [9], [10]), where the Complementary Side Information (C-SI) [2] is naturally available at both destinations.

A general multi-source and multi-relay scenario is analyzed in [11], where the WNC-related problems are approached by proposing the relays to decode only a certain linear functions of source data. The final destination then needs only to receive an adequate number of the “equations” (full rank observation) forwarded by relays to calculate all the data it desires. However, the problem of partial C-SI is not analyzed therein.

Although an extension of the WNC principles from the 2-WRC system into a slightly advanced network structures like the butterfly network [12] (Fig. 1) could seem relatively straightforward, several new unconventional features of the WNC arise. First of all, the unreliable transmission of the C-SI can be overcome by an increased cardinality of the relay output (extended cardinality relaying [13], [14]) – the phenomenon which does not have its counterpart in the traditional (binary) NC systems. The legacy of the NC approach is hence broken here, rising a huge number of new (and challenging) research problems.

It has been shown in [14] that the extended cardinality relaying/relay output mapping could (under specific channel conditions) outperform the minimal one (see e.g. [2]). This was demonstrated on an example of the butterfly network with 4-ary source and 8-ary relay constellation alphabets, using the Hierarchical Decode & Forward (HDF) strategy [2]. Moreover, as also proved in [14], the HDF strategy with a layered hierarchical coding approach (see e.g. [2]) and linear mapping can achieve the end-to-end capacity upper bound in the butterfly network.

B. Goal of this Paper and Contribution

One of the crucial steps in the HDF system design is the choice of a suitable Hierarchical Network Code (HNC) mapper at the relay. Simply speaking, this mapper defines how the separate data streams (from individual users) are mapped to the hierarchical (network-coded) data [2] stream at the relay node.

Design of the HNC mapper is quite simple in the case of the minimal mapping (see e.g. [2]) where it is usually given by a simple bit-wise xor operation. However, in case of the extended mapping the suitable HNC mapper should respect also the amount of destinations’ C-SI (i.e. a quality of the C-SI links) to maximize the potential throughput of the system.

In this paper we focus on this problem. We introduce a systematic approach to the design of a set of HNC mappers for relay output mapping in the Broadcast (BC) phase of communication in the butterfly network (Fig. 1). Consequently, we focus mainly on a relay processing and the subsequent BC phase of communication. The parametric Multiple-Access (MAC) phase (essentially identical to that in the 2-WRC) is not considered in this work.
analyzed e.g. in [2], [3], [15] and it is out of the scope of this paper.

In a numerical evaluation of the alphabet-constrained capacity (mutual information) we show how the performance of the butterfly network can be controlled by a suitable choice of the HNC mapper (from the pre-designed set) at the relay. These results pave the way for an idea of adaptive butterfly network, where the performance can be adapted to the actual channel conditions (and thus the quality of C-SI).

II. SYSTEM MODEL AND DEFINITIONS

A. Butterfly Network with HDF strategy

The butterfly network [13] comprises two (independent) data sources $S_A, S_B$ which want to transmit their data to the corresponding destinations $D_A, D_B$. Since the direct links $(S_A \rightarrow D_A$ and $S_B \rightarrow D_B$) are not available, assistance of the relay node $R$ is needed for a successful transmission of both data streams. We assume that the HDF strategy (see e.g. [2]) is employed in the system. Due to the half-duplex constraint each communication round consists of separate MAC and BC phases (see Fig. 1).

1) MAC phase: In the MAC phase both sources simultaneously transmit their signals to the relay. A signal space representation (with an orthonormal basis) of the transmitted channel symbols is $s_A(c_A), s_B(c_B)$ ($s_A, s_B \in \mathcal{A}_X \subset \mathbb{C}^N$, $|\mathcal{A}_X| = |\mathcal{A}_R| = |\mathcal{A}| = M$), where $c_A, c_B$ are source node code symbols, $\mathcal{A}_X$ is a channel symbol memoryless mapper and $|\mathcal{A}|$ is the source alphabet cardinality (assumed identical for both sources). The constellation space signal received at $R$ in MAC phase is

$$x = h_A s_A + h_B s_B + w_R,$$  

where $w_R$ is the circularly symmetric complex Gaussian noise (variance $\sigma^2_{w_R}$ per complex dimension) and $h_A, h_B$ are scalar complex channel coefficients (constant during the observation and known at the relay).

The parametric MAC phase has obviously a significant impact on the overall system performance (see e.g. [3], [14]), however this is beyond the scope of this paper. The main goal of this paper is the design of suitable HNC mappers (for arbitrary source alphabet and C-SI), and hence we assume a perfect MAC phase (reliable decoding of arbitrary function $c_{AB} = f(c_A, c_B)$ is possible at the relay). Nevertheless, the HNC mappers proposed in this paper do not depend on the particular source alphabets and hence they are feasible also for the butterfly network with optimized parametric source alphabets (i.e. Network Coded Modulation – see e.g. [3], [15]).

Sources’ transmissions in the MAC phase are overheard by $D_A, D_B$ (due to the inherent broadcast nature of wireless channels). These signals will be called as C-SI (see e.g. [2], [13]). Since each destination receives only the signal intended for the other destination (see Fig. 1), it is obvious that (on its own) the C-SI does not carry any information about the desired data. The constellation space signals received at $D_A, D_B$ (on the C-SI links) are

$$z_A = s_B + w_A,$$  

$$z_B = s_A + w_B.$$  

The C-SI links (2), (3) have a significant impact on the system performance (as will be discussed later). For the purpose of this paper we will evaluate the quality of the C-SI links by a number of “reliably” received channel symbol bits, i.e. $c^A_{CSI}$ (for $D_A$) and $c^B_{CSI}$ (for $D_B$), where $c^A_{CSI}, c^B_{CSI} \in \{0, 1, \ldots, \log_2 M\}$.

2) Relay processing: After receiving $x$, the relay performs an eXclusive mapping operation (see [2] for details) to map the received signal to the relay output symbol $s_R(c_{AB})$, where $c_{AB} \in \mathcal{X}^R(c_A, c_B)$. This hierarchical symbol mapping is referred to as the Hierarchical Network Code (HNC) map in [16]. In principle, the hierarchical (network-coded) output signal from the relay must jointly represent the data from both sources $(s_A, s_B)$, while a knowledge of respective C-SI at the destination is necessary for a successful decoding of the desired (separate) data stream [2].

For the purpose of this paper we will assume that the HNC mapping operation can be described directly on the signal space level $s_R^{k_{ij}}(s_A^{k_{ij}}, s_B^{k_{ij}})$. In this case it is suitable to describe the specific HNC mapping by the HNC matrix

$$X_{\text{HNC}} = \begin{bmatrix} k_{11} & \cdots & k_{1M} \\ \vdots & \ddots & \vdots \\ k_{M1} & \cdots & k_{MM} \end{bmatrix},$$  

where $k_{ij} = X_{\text{HNC}}(i, j), i, j \in \{1, 2, \ldots, M\}$ is an index of the relay output (signal space) channel symbol $s_R^{k_{ij}} \in \mathcal{A}_X^R$. Note
that for the cardinality of the relay output alphabet holds (see e.g. [2]):

\[
|\alpha_s| = M \leq |\alpha_s^R| \leq M^2 = |\alpha_s|^2.
\]  

(5)

The equalities in (5) correspond to the minimal (\(|\alpha_s^R| = |\alpha_s|\)) and full (\(|\alpha_s^R| = |\alpha_s|^2\)) mapping (see e.g. [2], [13]). More details about the relay output alphabet cardinality and corresponding required amount of the C-SI will be provided later.

3) BC phase: In the BC phase the relay transmits the signal to both destinations. The constellation space signals received at \(D_A, D_B\) are

\[
y_A = sR + wA, \quad (6)
\]

\[
y_B = sR + wB. \quad (7)
\]

Assuming that a suitable HNC mapper has been used at the relay, destinations are able to decode the desired data from the relay signal and C-SI.

B. SAPHYRE project point of view

The topology of the butterfly network (Fig. 1) can be justified also from the point of view of the SAPHYRE project [17] (see Fig. 2). Sources \(S_A, S_B\) correspond to the base stations of two different mobile operators. Since both mobile nodes (destinations \(D_A, D_B\)) are out of their respective operator’s coverage area, the communication is not possible. However, if the operators implement the SAPHYRE approach, which in this case means that they will share their time-frequency slots (sharing resources) and deploy a common relay \(R\) (sharing infrastructure), they will both benefit from the situation, since a communication to both mobile nodes will be made possible (using the HDF strategy).

III. THE ROLE OF COMPLEMENTARY SIDE INFORMATION

The amount of information on the complementary data stream (C-SI) which is available at \(D_i\) (after the MAC phase) is given by a number of reliably received bits \(c_{CSI}^i, i \in \{A,B\}\). This corresponds to a level of granularity at which the destination can distinguish particular symbols received on the C-SI link, and consequently also to the partitioning of the HNC map (HNC matrix \(X_{HNC}\)). For the unambiguous decoding each destination needs to identify only the relay symbol inside the resulting subset of the HNC map (submatrix of \(X_{HNC}\)). This is obvious from the example system with \(|\alpha_s| = 4\) in Figs. 3, 4, 5 for perfect, zero and partial C-SI (respectively).

As proved e.g. in [14], for unambiguous decoding in the presence of perfect C-SI at each destination each particular relay output symbol can appear at most once on each row and in each column of the HNC map (see Fig. 3). This corresponds to the minimal cardinality relay output mapping (\(|\alpha_s^R| = M\)). Similarly (as also noted in [14]), for the case of partial/imperfect C-SI each particular relay output symbol can appear at most once within a group of rows/columns (given by the HNC map partitioning – see Fig. 5). This corresponds to the extended cardinality relay output mapping. In the last case, i.e. in the butterfly network where the C-SI links are fully unreliable (Fig. 4), the relay needs to deliver data to both sources without assistance of the C-SI and hence a full cardinality relay output mapping (\(|\alpha_s^R| = M^2\)) is required.

As it is obvious from the discussion above, the C-SI (overheard by destinations in the MAC phase) has a crucial impact on the overall system performance, since it determines a required minimal cardinality of the relay output alphabet \(|\alpha_s^R|\) (as proved e.g. in [14]). This is obviously a direct consequence of the eXclusive law (see e.g. [2]). Accordingly, the C-SI introduces some preconditions on a suitable HNC matrix design. In the following section we introduce an algorithm for the design of a set of HNC matrices for arbitrary source alphabet cardinality \(M\) (and the corresponding range of permissible values of \(c_{CSI}^A, c_{CSI}^B\)).

IV. HNC MAPPER DESIGN

As noted in the previous section, the quality of the C-SI links defines the partitioning of the HNC matrix \(X_{HNC}\) (see examples in Figs. 3, 4, 5), where unique relay output
one C-SI link is fully unreliable (e.g. a weaker C-SI link). To clarify this statement we can consider an example where we use full cardinality relaying in order to guarantee successful decoding at the destination (even if the destination is unable to distinguish between these two symbols on the C-SI link. Hence, these symbols should be grouped into the same subset of the HNC map and consequently the successful decoding would be possible even if the destination is unable to distinguish between these two symbols on the C-SI link.

The complete set of HNC matrices $X_M^{(c_{CSI})}$ for given $M$ and all permissible values of $c_{CSI}$ (0 ≤ $c_{CSI}$ ≤ log2$M$) can be designed with Algorithm 1. This design algorithm is suitable even for the design of HNC mappers for minimal ($\mathcal{B} = 1$, $L = M$) and full cardinality ($\mathcal{B} = M$, $L = 1$) relaying. Some examples of the proposed HNC matrices are given in Tables I, II for $M = 4$ and $M = 8$ (respectively). As it is also obvious from these tables, the cardinality of the relay output alphabet is given by

$$|\mathcal{A}_R| = |\mathcal{X}| = \frac{M^2}{L},$$

(8)

A. Source Alphabet Partitioning

The HNC mapper design in Algorithm 1 is based on the assumption that each destination can identify (at least) first $c_{CSI}$ bits from the constellation symbol received on the C-SI link (except for the full cardinality case) – see examples in Figs. 3, 5. Obviously, it is possible to increase the reliability of this (partial) C-SI by a suitable partitioning (and corresponding constellation indexing) of both source alphabets $\mathcal{A}_s$. The idea is to partition the source constellation alphabet into smaller subsets (according to the principles similar to Ungerboeck mapping rules [18]) to increase the probability of successful (partial) C-SI retrieval at the destination. Hence the intention is to maximize the distance between the particular subsets (not the particular symbols). The principle of the pro-

1The minimal required relay alphabet cardinality $|\mathcal{A}_R|$ is given by the weaker C-SI link. To clarify this statement we can consider an example where one C-SI link is fully unreliable (e.g. $c_{CSI}^A = 0$). In this case the relay must use full cardinality relaying in order to guarantee successful decoding at $D_A$ (even if $D_A$ has a perfect C-SI available). Hence for the purpose of the HNC mapper design we can assume the same level of partitioning given by $c_{CSI} = \min\{c_{CSI}^A, c_{CSI}^B\}$, resulting in the square blocks $B_1$.

2This should be clear e.g. from Fig. 5, where symbols 00 and 01 define the same subset of the HNC map and consequently the successful decoding would be possible even if the destination is unable to distinguish between these two symbols on the C-SI link. Hence, these symbols should be grouped into one subset $\{00, 01\}$ (and similarly for the other symbols).
The pure C-SI link capacity with maximum rate given by the broadcast phase capacity region is rectangular. Without loss of generality, we focus on the proposed HNC mapper design (Algorithm 1) for the butterfly network. The achievable broadcast phase capacity of this (genie-aided) system is evaluated in Figs. 19, 20 for QPSK and 8PSK (respectively).

V. NUMERICAL EVALUATIONS

In this section we finally demonstrate a feasibility of the proposed HNC mapper design (Algorithm 1) for the butterfly network. Without loss of generality, we focus on the $D_B$ processing. Destination $D_B$ has two channel observations—the relay (network-coded) signal $y_B|c_B$ and C-SI link $z_B|c_A$.

Since the relay transmits a common signal to both destinations, the broadcast phase capacity region is rectangular [13] with maximum rate given by $I(c_B;y_B,z_B)$ (and similarly for $D_A$). The broadcast phase alphabet-constrained capacity [13] (mutual information) is given by

$$C_{HBC} = I(c_B;y_B,z_B) = H[y_B,z_B] - H[y_B|c_B].$$

The pure C-SI link capacity $C_{CSI} = I(c_A;z_B)$ will be also evaluated (for a comparison)\(^3\). This capacity is simply a capacity of the Gaussian (alphabet constrained) channel [13].

\(^3\)It is important to note that some caution is necessary when interpreting the numerically evaluated C-SI link capacity and the C-SI link “quality indicator” $c_{CSI}$ used in this paper. Obviously, the particular value of $c_{CSI}$ does not directly correspond to the C-SI link capacity and the proper relation between these quantities needs further investigation.

The alphabet constrained capacities were evaluated numerically by the Monte-Carlo integral evaluation (for details see [13] and references therein). The broadcast phase capacity was evaluated as a function of the C-SI link SNR ($\gamma_B$) for various relay-to-destination SNR ($\gamma_D$). The results are on Figs. 9, 10, 11 (QPSK source alphabet) and on Figs. 13, 14, 15, 17 (8PSK source alphabet). For all these results the source alphabets were partitioned according to the Figs. 7, 8 and the HNC mappers (matrices $X^{(CSI)}_M$) were designed using the Algorithm 1.

For the extended relaying scenarios we have evaluated also the capacities for source alphabets with Ungerboeck [18] mapping (see Figs. 12, 16, 18). Here a significant degradation of the broadcast phase capacity can be observed (see Figs. 11 and 12; Figs. 15 and 16; Figs. 17 and 18), although the C-SI link capacity is identical for both compared source alphabet partitioning. Hence it is obvious that Ungerboeck source alphabet partitioning is not appropriate for the proposed HNC mappers.

A. Adaptive Butterfly Network Performance

As it is clear from the presented numerical results, the achievable broadcast phase capacity depends strongly on the employed HNC mapper. Obviously, by a suitable choice of the HNC mapper (according to the actual channel conditions) it would be possible to maximize the throughput of the system. Provided that the observed quality of the C-SI link is delivered to the relay by both destinations $D_A, D_B$, a suitable HNC mapper can be chosen (from the predesigned set) to maximize the achievable capacity of the network.

To demonstrate a potential performance of such a system, we will assume a genie-aided relay which possess a perfect knowledge about the actual quality of the C-SI links quality ($\gamma_A, \gamma_B$) and the respective relay to destination links ($\gamma_{DA}, \gamma_{DB}$). The achievable broadcast phase capacity of this (genie-aided) system is evaluated in Figs. 19, 20 for QPSK and 8PSK (respectively) source alphabets.

VI. CONCLUSION

In this paper we have proposed a systematic algorithm for the HNC mapper design (see Algorithm 1) for relaying in the butterfly network with HDF strategy. The feasibility of the proposed HNC mappers was demonstrated by evaluation of the
H-BC and C-SI capacity for alphabets $\mathcal{A}_s$: QPSK and $\mathcal{A}_R$: QPSK.

H-BC and C-SI capacity for alphabets $\mathcal{A}_s$: QPSK-Ungerboeck and $\mathcal{A}_R$: 8PSK.

H-BC and C-SI capacity for alphabets $\mathcal{A}_s$: 8PSK and $\mathcal{A}_R$: 8PSK.

H-BC and C-SI capacity for alphabets $\mathcal{A}_s$: 8PSK and $\mathcal{A}_R$: 16QAM.

H-BC and C-SI capacity for alphabets $\mathcal{A}_s$: 8PSK and $\mathcal{A}_R$: 64QAM.

H-BC and C-SI capacity for alphabets $\mathcal{A}_s$: QPSK and $\mathcal{A}_R$: QPSK.

H-BC and C-SI capacity for alphabets $\mathcal{A}_s$: QPSK-Ungerboeck and $\mathcal{A}_R$: 8PSK. Compare with Fig. 11.

H-BC and C-SI capacity for alphabets $\mathcal{A}_s$: QPSK and $\mathcal{A}_R$: 16QAM.
Figure 15. BC phase alphabet constrained capacity and capacity of the complementary SI link (extended mapping, $A_s = 8$PSK, $A_R = 16$QAM).

Figure 16. BC phase alphabet constrained capacity and capacity of the complementary SI link (extended mapping, $A_s = 8$PSK with Ungerboeck mapping, $A_R = 32$QAM). Compare with Fig. 15.

Figure 17. BC phase alphabet constrained capacity and capacity of the complementary SI link (extended mapping, $A_s = 8$PSK, $A_R = 32$QAM). Compare with Fig. 16.

Figure 18. BC phase alphabet constrained capacity and capacity of the complementary SI link (extended mapping, $A_s = 8$PSK with Ungerboeck mapping, $A_R = 32$QAM). Compare with Fig. 17.

Figure 19. Maximum achievable BC phase (alphabet constrained) capacity for the adaptive butterfly network ($A_s = QPSK$).

broadcast phase capacity. We have shown that suitable source alphabet partitioning should be performed by the sources to increase the probability of successful (partial) C-SI retrieval at destinations (to maximize the achievable capacity of the system). The observed behaviour of the broadcast phase capacities gave rise to the idea of adaptive butterfly network, where the performance of the system can be adapted to the actual quality of the C-SI and relay to destination links.

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Figure 20. Maximum achievable BC phase (alphabet constrained) capacity for the adaptive butterfly network ($\alpha_s = 8$PSK).


